

Many-Body Chaos in Fully Developed Turbulence

A THESIS

submitted in partial fulfilment of the requirements for the degree of
Integrated Bachelor of Science (BS) and Master of Science (MS).

Aikya Banerjee
(ঐক্য ব্যানার্জী)



Department of Physical Sciences,
Indian Institute of Science Education and Research Kolkata,
Mohanpur - 741246, India

guided by

Prof. Samriddhi Sankar Ray



International Centre for Theoretical Sciences,
Tata Institute of Fundamental Research, Bengaluru - 560089, India

Date April 30, 2026

To my father late Samares Bandyopadhyay

Declaration

The work presented in this thesis, titled “Many-Body Chaos in Fully Developed Turbulence”, is original and was carried out by me as an integrated Bachelor of Science (BS) and Master of Science (MS) student at the Department of Physical Sciences, Indian Institute of Science Education and Research, Kolkata under the supervision of Dr. Samriddhi Sankar Ray at the International Centre for Theoretical Sciences, Bengaluru.

Parts of the thesis, particularly those dealing with the direct numerical simulations of the Navier-Stokes equation and associated theory were carried out in close collaboration with Ritwik Mukherjee, a graduate student at the International Center for Theoretical Sciences, Bengaluru.

(Aikya Banerjee)

Department of Physical Sciences,
Indian Institute of Science Education and Research,
Kolkata - 741246, India

Date: April 30, 2026

Certificate

This is to certify that the work presented in this thesis, titled “Many-Body Chaos in Fully Developed Turbulence”, submitted by Aikya Banerjee (Roll No: 20MS104) to the Indian Institute of Science Education and Research, Kolkata, in partial fulfilment of the requirements for the Bachelor of Science - Master of Science (BS-MS) degree in Physical Sciences, has been carried out by him under my supervision.

To the best of my knowledge, this work is original.

(Prof. Samriddhi Sankar Ray)

International Centre for Theoretical Sciences,

Tata Institute of Fundamental Research,

Bengaluru - 560089, India

Date: April 30, 2026

Acknowledgement

Achievements and setbacks are important milestones in life, but it is the journey that truly shapes who we are. Along the way, we encounter people who guide us, support us, and make our experiences truly meaningful. Their encouragement and wisdom leave a lasting impact, making every step worthwhile. It gives me immense pleasure to thank everyone who has contributed to my growth, both academically and personally, and whose presence has made this journey a memorable and enriching one.

I want to begin by thanking my master's thesis advisor, Prof. Samriddhi Sankar Ray, who has not only guided me academically but also mentored me in various aspects of life. That he helped me frame my research interests and direct my goals would be an understatement. Beyond research, I am deeply indebted to him for his invaluable guidance, which has undoubtedly helped me become more composed and organized. I am profoundly grateful for his unwavering support and guidance. I also want to extend my heartfelt thanks to his wonderful family—Purbasha Di, Mrittika, and Mrinalini. Every time I met them, I felt at home.

During my undergraduate years, I was fortunate to have Professor Pradeep Kumar Mohanty as my mentor. From analytical techniques to numerical schemes, he patiently taught me everything step by step. Each conversation with Prof. Mohanty helped me grow and deepened my interest in the subject. Beyond academics, he also looked out for me like a guardian. His unwavering support and profound kindness always provided me with a sense of comfort. He taught me to cherish every single attempt at understanding scientific problems, regardless of success, as each effort contributes to growth and learning. My debt to him cannot be measured.

I would like to express my gratitude to Professor Axel Brandenburg. The brief but insightful discussions I had with Axel remain crucial to my undergraduate research experiences. His boundless energy and spontaneous curiosity instilled in me a deep appreciation for research. I am also deeply grateful to Professor Peter Virnau, under whose guidance I explored a new direction in statistical mechanics, which helped me to strengthen my computational skills. I would also like to thank Professor Nishant Singh. The analytical and numerical skills that I learned from him were instrumental for me throughout the journey.

I am also thankful to the faculty members at ICTS, especially Prof. Subhro Bhattacharjee, Prof. Sthitadhi Roy, Prof. Deepak Dhar, and Prof. Anupam

Kundu. I am also grateful to Prof. Jeremie Bec for the helpful discussions. I would like to thank the Long Term Visiting Student Programme (LTVSP) at ICTS for enabling me to conduct my master's thesis research there.

I want to express my heartfelt gratitude to all the faculty members at the Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata. From helping me frame my research interests to nurturing and strengthening my foundation, their unwavering support and guidance have been crucial. I am grateful to the Department of Science and Technology, Government of India, for the Kishore Vaigyanik Protsahan Yojana (KVPY) fellowship, which made my education possible.

In our journey, there are a few people who guide us and show us the path. However, the people who walk with us down the road are also an integral part of the journey. Their presence often makes the journey memorable and enjoyable. Ritwik Mukherjee (Ritwik Da) not only guided me academically but also made my stay at ICTS memorable in every aspect. Apart from Physics, we often discussed life, culture, movies, cooking and many other things. I am also thankful to Prof. Siddhartha Mukherjee for his insightful remarks. I would also like to thank Mrinal, Anikat Da, Prateek and other members of the fluid group at ICTS.

I am deeply indebted to Soumya K. Saha, Indranil Mukherjee, and Sayantan Mitra for insightful discussions. It has been a pleasure for me to share thoughts with such joyful and cheering seniors. I am thankful to Priyajit Jana for his continuous support. We not only shared rooms for a couple of years, but we also shared ideas, Physics perspectives and thoughts. I am also grateful to Saminur, Sreejit, Sreetama, Sayantan, Rounak, Swapnil and Anish for making my college life joyful. I am also thankful to Tisha Dash. She has not only supported me in the last couple of years but also inspired and cheered me.

I am thankful to Nishant Uchale, my roommate and friend at ICTS. Nishant supported me through all the ups and downs of my thesis year. I am also grateful to Ashif, Bikram Da, Sayantan, Swadesh, Shreyash, Subhadeep, Ramnath, Shivani, Shradhanjali, Preethi, Devesh, Saptarshi Da, Vinay, Santhiya and all of my other friends at ICTS.

Last but not least, I want to thank my parents. My father, Samares Bandyopadhyay, has always been my inspiration since childhood. His dedication to fostering education instilled in me a great fondness for basic science. Despite his untimely demise, he always inspired and cheered me to face all the obstacles. I am also deeply indebted to my mother, Rekha Banerjee. Starting from teaching and guiding me, she protected me during all the turbulent phases of my life. I am also thankful to all other family members, Mitra Pisi (Mitra Bhattacharjee), Sanjay Jethu (Dr Sanjay Dhar) and Sanjib Kumar De, for their constant guidance and support.

Contents

I	Introduction	8
II	Shell model of turbulence	13
II.1	The early days	14
II.2	GOY shell model	16
II.3	Numerical Simulations	17
III	Many-body chaos in shell models	19
III.1	Decorrelator results	19
III.2	Numerical schemes for decorrelator in shell models	21
III.3	Analytical expression	22
IV	Many-body chaos in the Navier-Stokes equation	24
V	Decorrelator from the Navier-Stokes equation and its limits	30
VI	Conclusions	33
VII	Appendix	36
VII.1	Direct Numerical Simulation of the N.S. equation	36
VII.2	Numerical Integration Schemes	37

I Introduction

The nature of many-body chaos in fully developed turbulence is perhaps one of the most celebrated question in last few decades. Turbulent flows stand out from other instances of classical many-body systems exhibiting chaotic behaviour because they are ubiquitous in the natural world. Thus, it should come as no surprise that physicists working on problems in oceanography, atmospheric sciences, astrophysics, and fluid dynamics must take into account the fundamentally chaotic character of such flows [1]. This prevalence is not unexpected, as the fundamental Navier-Stokes equation in all of these systems leads to turbulent solutions in the limit of small viscosities [2].

While mathematical and engineering tools remain indispensable, the ideas of statistical physics provide the basis for much of our understanding of turbulence, especially in its most general form where the flow is homogeneous and isotropic. In particular, chaos along with the other fingerprint of fully developed turbulence, namely intermittency, and their dependence *inter alia* the Reynolds number Re , forms a major challenge in a complete understanding of turbulence [2]. Thus, this continues to be a fundamental problem in the areas of statistical physics of non-equilibrium, driven-dissipative systems and for many the holy grail in fully developed turbulence.

Butterfly effect is a signature of many-body chaos in dynamical systems. As alluded in the following poem describing the cascade of events, butterfly effect is characterised by the amplification of microscopic fluctuation leading to large scale structure formation.

“For want of a nail, the shoe was lost.

For want of a shoe, the horse was lost.

For want of a horse, the rider was lost.

For want of a rider, the battle was lost.

For want of a battle, the kingdom was lost!" [3]

In characterising chaos in classical dynamical systems, the idea of butterfly effect is very crucial. The enhancement of the initial perturbation helps in quantifying the extent of chaos by estimating what we call as the Lyapunov exponent. In general the Lyapunov exponent is estimated by taking two nearby trajectories in the phase space and measuring how they start separating out with time. If the initial perturbation is δx_0 and at time t the separation is $\delta x(t)$, then the Lyapunov exponent (λ) is measured as:

$$|\delta x(t)| \approx e^{\lambda t} |\delta x_0| \quad (1)$$

If the value of λ is positive then the perturbation grows exponentially leading to chaotic behaviour. The more positive the value of λ is the more the system is chaotic.

In fully developed turbulence the first characterization of chaos was done by Ruelle [4] following dimensional arguments and the Kolmogorov's seminal work from 1941 [5]. Ruelle showed that the largest Lyapunov exponents $\lambda \sim Re^{0.5}$ by relating it to the inverse of the eddy-turnover time τ_η related to the smallest length scale of the problem.

$$\lambda = \lim_{t \rightarrow 0} \frac{1}{t} \ln \left| \frac{\delta \psi(t)}{\delta \psi(0)} \right| \sim \frac{\bar{\epsilon}^{1/3}}{\eta^{2/3}} \sim \frac{\bar{\epsilon}^{1/2}}{\nu^{1/2}} \sim \frac{1}{\tau_\eta} \quad (2)$$

where $\delta \psi$ is the disturbance, $\bar{\epsilon}$ is the average energy dissipation, ν is the kinematic viscosity and η is the Kolmogorov dissipation length scale. Here we used the fact that the Kolmogorov dissipation length scales as $\eta \sim (\nu^3/\bar{\epsilon})^{1/4}$.

The validity of the Ruelle's calculations relies on the existence of a unique

scaling law (*monofractal* limit) and unique small length and time scale as alluded in the Kolmogorov picture of turbulence. However, experimental, numerical and observational evidence against this is ubiquitous. A modern realisation of turbulence as described by Frisch and Parisi [6] relies on a multifractal approach. This leads to correction to the Reynolds dependence of the Lyapunov exponent. From the Kolmogorov theory one predicts that the dissipation length η scales as $L/Re^{3/4}$ with L as the integral length scale. Using this form of η we thus get $\lambda \sim Re^{1/2}/T_0$. In the multifractal picture, for an eddy of size $l(=r/L)$ if h is the Hölder exponent of the velocity difference (δu_l) such that:

$$\delta u_l = |u(x+r) - u(x)| \sim u_0 l^h$$

The integral scale Reynold's number is defined as $Re \sim Lu_0/\nu \sim L^{1+h}/\nu$ leading to the following relation: $\nu \sim L^{1+h}/Re$. In the dissipation scale, $u_\eta \eta (\sim \eta^{1+h}) \sim \nu$. Therefore, by combining these two cases we have: $\eta \sim LRe^{-1/(1+h)}$. Since, the Lyapunov exponent scales as $\lambda \sim 1/\tau_\eta \sim u_\eta/\eta \sim \eta^{h-1} \sim Re^\alpha$, it can thus be written as:

$$\alpha = \frac{1-h}{1+h} \quad (3)$$

In the Kolmogorov picture of turbulence $h = 1/3$ is uniquely defined. Plugging this in Eq 3, one gets back the result predicted by Ruelle, $\lambda \sim \sqrt{Re}$.

Vulpiani et al. [7] exploited this multifractal correction for the exponent α using the random beta model for turbulence [8]. They calculated $\alpha = 0.459$ which is less than what Ruelle has argued following the Kolmogorov model for turbulence. In an analogous study for a model of turbulence namely the shell model, they have shown that the largest Lyapunov exponent scales as $Re^{0.459}$.

Early characterisation of Lyapunov exponent and the Lyapunov spectra was

carried out by Yamada and Ohkitani [9] in the shell models for turbulence. They have further highlighted that the largest Lyapunov exponent scales as $\nu^{-1/2}$. Following their footsteps, Vulpiani et al [7] showed that the average value of the largest Lyapunov scales as $\lambda_1 \sim \text{Re}^{0.459}$. However, for a model of turbulence, such method of averaging is questionable [10, 11] and essentially does not account for the effect of intermittency properly. In recent years, Ho et al [12] claimed using direct numerical simulations that $\alpha = 0.53$.

The lack of a coherent picture between the theoretical estimate and measurements — as well as the lack of agreement between different simulations [7, 12, 9, 13–17] — suggests that the origins of chaos in turbulence is far from settled.

We take a different path. In recent years, the generalisation of the out-of-time-order commutator (OTOC) of quantum many-body systems to classical many-body systems opened up new directions towards understanding a few fundamental aspects of statistical physics, namely ergodicity and thermalization [18]. This analogous version of OTOC namely the decorrelators [18] are extensively studied in classical spin chains [19, 20] opening a different route. Recently, this idea has been extended for thermalised flows [21], especially in Galerkin truncated Euler and one dimensional Burger equation. In such systems, it has been seen theoretically and numerically that the Lyapunov exponent characterising the decorrelator growth scales with temperature as $\lambda \sim \sqrt{T}$. Despite the similarity in temperature scaling, there is a contrasting difference between such systems. While the local nature of the spin systems preserves a clear surface of growth, incompressible thermalised flows due to non-local nature lacks a pilot wave for perturbation spread. This non-locality also plays a crucial role in the Navier-Stokes equation. However, it is to be noted that the

driven-dissipative nature of such systems makes these fundamentally different from the thermalised flows.

Thus, an extension of the idea of the decorrelators appears as a suitable and interesting candidate towards understanding the nature of many-body chaos in fully developed turbulence. I study this problem in two different cases. I start with a model of turbulence (easy to tract numerically) to set the stage for the direct numerical simulation of the Navier-Stokes equation (the end part of the work).

II Shell model of turbulence

*“Big whirls have little whirls
that feed on their velocity,
And little whirls have lesser whirls
and so on to viscosity.” [22]*

We will start our discussion with a cascade model of turbulence namely the shell model. For the Navier-Stokes equation in three dimensions, the resolution for numerical simulation largely depends on the Raynold’s number of the system. Typically, in the mean field Kolmogorov picture of turbulence [5], the viscous dissipation scale η scales as $\text{Re}^{-3/4}$. Thus for three dimensional Navier Stokes equation with changing Re one should be careful of the numerical resolution of the system, which should in principle scale as $1/\eta^3 \sim \text{Re}^{9/4}$ [23]. Need for such a large resolution makes numerical simulations for large Re very difficult. Therefore, a simpler yet phenomenologically similar model of turbulence emerged which we now call as the shell model for turbulence.

The earlier works on shells models by reducing the number of degrees of freedom date back to the 1970’s [24–26]. The idea was to divide the wave numbers into concentric shells such that for n th shell the radius is $k_n = k_0 \lambda_n$ [23] with $\lambda(> 1)$ as the shell parameter. In this way the n th shell contains $(4\pi k_0^3/3)\lambda^{3n}(\lambda - 1)$. This essentially reduces the number of degrees of freedom by a large margin. The triadic interaction due to nonlinearity and the pressure terms are adjusted by introducing a dummy non-linear term G_n [23]. Indeed, the full Navier-Stokes equation in three dimensional periodic grid is simplified as —

$$\left(\frac{d}{dt} + \nu k_n^2 \right) = G_n[u, u] + f_n \quad (4)$$

But fully developed turbulence possesses a rich phenomenological structure. How a toy model can really mimic such a system? The central theme of this section is to highlight some of the key aspects of the shell model and what makes it a ideal model for fully developed turbulence. One of the major aspect of using the shell model is that it is dimensionally consistent with the spectral Navier-Stokes equation. If the nonlinear term is tweaked optimally it mimics the cascade structure of fully developed turbulence.

Furthermore, over time the shell model depicts the effect of intermittency nicely. Over time the energy dissipation rate shows a strong intermittent behavior, a fundamental feature of fully developed turbulence.

On the other hand, as the velocities in each shells are coarse grained and shrunk into a one dimensional shell-space, it does not account any effect of geometry and therefore, these models are irrelevant in understanding structures in different scales in turbulent flows.

II.1 The early days

The shell model for turbulence started off its journey with Obukhov's work [27] from the perspective of atmospheric sciences. However, the structural similarity and the identical energy cascade embraced the seed to the onset of the shell models. He proposed a set of first order nonlinear ordinary differential equations with discrete wave numbers k_n . The equation for shell numbers n is written in the following way:

$$\frac{du_n}{dt} = a_{n-1}u_{n-1}u_n - a_nu_{n+1}^2 - \nu u_n\delta_{n>N} + f\delta_{n,1} \quad (5)$$

The first two terms in the right hand side of the equation are the non-linear terms. The third term or the dissipative term acts at large shell numbers

($> N$). The last term is a forcing term relevant in the smallest wave number. Phenomenologically, this equation has large scale energy injection (small shell number) and small scale dissipation (large shell number) and it is quite close to the Navier-Stokes equation. In the inviscid limit that is if $f = \nu = 0$, the equation ensures that the total energy $E_{\text{tot}} = \sum_n u_n^2/2$ remains constant. By multiplying u_n to equation 5 in the inviscid and unforced limit, and then summing over all indices, one gets the relevant energy conservation:

$$\frac{dE}{dt} = \sum_n a_{n-1} u_n^2 u_{n-1} - \sum_n a_n u_n u_{n+1}^2 = 0 \quad (6)$$

However, the major problem with this model is that it does not satisfy Liouville's theorem in inviscid and unforced limit [28]. It is trivial to show that this is indeed true. If $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$ are the generalised co-ordinates for a dynamical system (applicable to systems that follow Hamiltonian equation), the Liouville's theorem ensures that the flow in phase space is incompressible (in the phase space co-ordinate) that is $\partial y_i / \partial y_i = 0$ [28, 29]. However, the Obukhov equation does not satisfy the criterion for Liouville's theorem. It is to be noted that for the Navier-Stokes equation if we plug in $\nu = \mathbf{f} = 0$, it satisfies the criterion for Liouville's theorem.

A modified equation was proposed by Gledzer [26] which ensures that the phase space flow is incompressible.

$$\frac{du_n}{dt} = A_n u_{n+1} u_{n+2} + B_n u_{n-1} u_{n+1} + C_n u_{n-2} u_{n-1} - \nu_n u_n \delta_{n>n_d} + f_n \quad (7)$$

with a boundary condition $u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0$.

II.2 GOY shell model

A more recent version of the Gledzer shell model, frequently known as the GOY model (named after Gledzer, Okhitani and Yamada [26, 30]) is defined over shells $k_n = k_0 \lambda^n$ with complex shell velocities u_n as —

$$\left(\frac{d}{dt} + \nu k_n^2 \right) u_n = -\iota k_n (u_{n+1} u_{n+2} - \frac{\epsilon}{\lambda} u_{n-1} u_{n+1} + \frac{\epsilon - 1}{\lambda^2} u_{n-2} u_{n-2})^* - \nu k_n^2 + f_n \quad (8)$$

where λ is the shell spacing and ϵ is derived from the relevant conservation laws. But how do we come from generic prefactors A_n , B_n and C_n to equation 8? For the systematic development of equation 8, we need to exploit the symmetry of the Navier Stokes equation. The Navier-Stokes equation remain invariant under the transformation $(t, k, u(k), \nu) \rightarrow (\varsigma^{1-h} t, \varsigma^{-1} k, \varsigma^{D+h} u(\varsigma^{-1} k), \varsigma^{1+h} \nu)$ where D is the dimensionality of the system. Does a similar conservation law holds for equation 7? We first consider $A_n = a_n k_n$, $B_n = b_n k_n$, and $C_n = c_n k_n$ in equation 7.

Let us now check the validity of scaling law by plugging an analogous scaling form: $(t, k_n, u_n) \rightarrow (\varsigma^{1-h} t, k_m, \varsigma^h u_m)$ where $k_m = k_n / \varsigma$ leading to a rescaled nonlinear term $\tilde{\Pi}_m^{\text{non-lin}} = k_m (a_n u_{m+1} u_{m+2} + b_n u_{m+1} u_{m-1} + c_n u_{m-2} u_{m-1})$. If the system is invariant under such a scaling the rescaled non-linear term should match the original form. Hence, $a_n = \tilde{a}$, $b_n = \tilde{b}$ and $c_n = \tilde{c}$ are independent of n .

For the no-force inviscid case, the energy should remain constant. Multiplying u_n^* and putting the $\dot{E}_n$ condition one obtains:

$$\sum_n k_n \left(\tilde{a} u_n u_{n+1} u_{n+2} + \tilde{b} u_{n-1} u_n u_{n+1} + \tilde{c} u_{n-2} u_{n-1} u_n \right)^* = 0$$

Therefore, by plugging in the boundary condition one obtains:

$$k_n(\tilde{a} + \lambda\tilde{b} + \lambda^2\tilde{c}) = 0$$

Let us assume $\tilde{a} = a$, $\tilde{b}\lambda = b$ and $\tilde{c}\lambda = c$. Using these new set of variables the previous condition can thus be written as — $a + b + c = 0$. We typically write $a = 1$, $b = -\epsilon$, and $c = (\epsilon - 1)$ where the value of ϵ is determined from the conserved quantities. This leads to the form of equation 8. In three dimensions, the typical choice is $\epsilon = 1/2$ and $\lambda = 2$.

There are different versions of the shell model (e.g. Sabra shell model). However, in this work, all my results are centered around the GOY model. To study the relevance of the GOY model and understand its characteristic similarity with the Navier-Stokes equation, I present numerical simulation of shell model in the next part of this thesis.

II.3 Numerical Simulations

The relevance of the shell model unfolds through the numerical simulations of the model. In equation 8 we unfold the general structure of the GOY model where we have contributions from the next and next-to-neaxt nearest neighbours. The shell variable u_n is defined on each shell and is a complex variable. The shell parameter $\lambda = 2$ so that the spectral modes are defined as $k_n = k_0 2^n$. As described above, such a choice lead to the constraint that $\epsilon = 0.5$. Initially we define $u_n(t = 0)$ with random phase and a modulus $|u_n| = \epsilon \exp(-k_n)$. Given the initial condition we evolve the system following equation 8 with a fourth order Runge-Kutta scheme. The details of the numerical schemes are discussed later in Appendix.

The force is applied to the first shell with an amplitude $0.1\delta_{n2}$. We choose

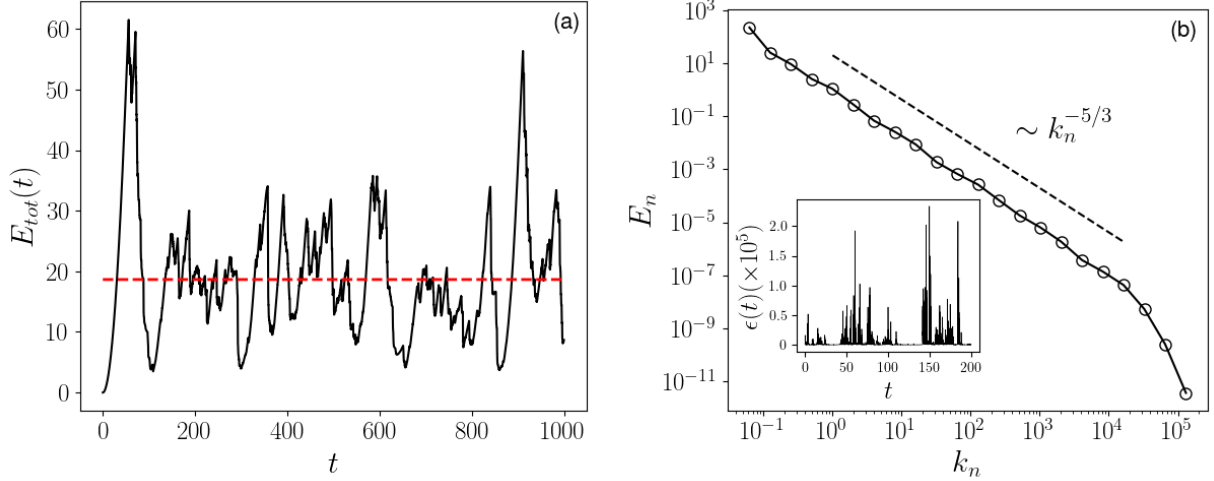


Figure 1: (a) In the left panel the evolution of the total energy as a function of time is shown. The system reaches a steady state near $t = 200$. The total energy remains roughly the same which indicates that on an average the energy injection by large scale forcing is balanced by the effect of dissipation. The red dashed line is the average value total energy. (b) In the right panel, the energy spectra $E_n = |u_n|^2/k_n$ is shown. For the GOY shell model the energy scales as $E_n \sim k_n^{-5/3}$. A reference line (dashed) is added to illustrate the scaling. In the inset, the rate of energy dissipation is plotted against time. The intermittent dissipation rate highlights a behavioural similarity between the shell model and the Navier-Stokes equation.

$\nu = 2 \times 10^{-7}$ and total number of shells $N = 22$. Additionally, the boundary condition is imposed in the following way: $u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0$.

In the left panel of Fig 1, I have shown how the system reaches the steady state as the total energy roughly reaches a plateau. In the steady state the shell-wise energy also shows a Richardson like cascade with a Kolmogorov like $k_n^{-5/3}$ scaling. Fully developed turbulence shows this $k^{-5/3}$ scaling of energy cascade. An analogous result in the shell model highlights the importance and structural similarity of the shell model to the full Navier-Stokes equation. In the right panel of Fig 1, we illustrate the numerically obtained energy spectra for the shell model and show that it scales as $k_n^{-5/3}$. In the inset, the energy dissipation rate is shown over a large period of time, the intermittent nature of the energy dissipation rate $\sum_n k_n |u_n|^2$ highlights the phenomenological similarity of the shell model to the Navier-Stokes equation.

III Many-body chaos in shell models

Let us begin our study of many-body chaos for fully developed turbulence with the simple case of the shell model which is easy to tract numerically. The phenomenological energy cascade and the intermittent energy scaling makes it an ideal starting point to explore the decorrelator problem in turbulent flows.

III.1 Decorrelator results

As discussed in the last section, in the Gledzer-Ohkitani-Yamada (GOY) model [26, 30], the velocities u_n , associated with a scalar, exponentially growing, wave-number $k_n = k_0 2^n$, are treated as complex, dynamical variables, with the shells numbers being integers $1 \leq n \leq N$.

In the statistically steady state, let the velocity *field* be denoted as u_n^A . We now construct a second field $u_n^B = u_n^A \forall n \neq n_p$ and $u_{n_p}^B = (1 + \epsilon_0)u_{n_p}^A$. Our choice

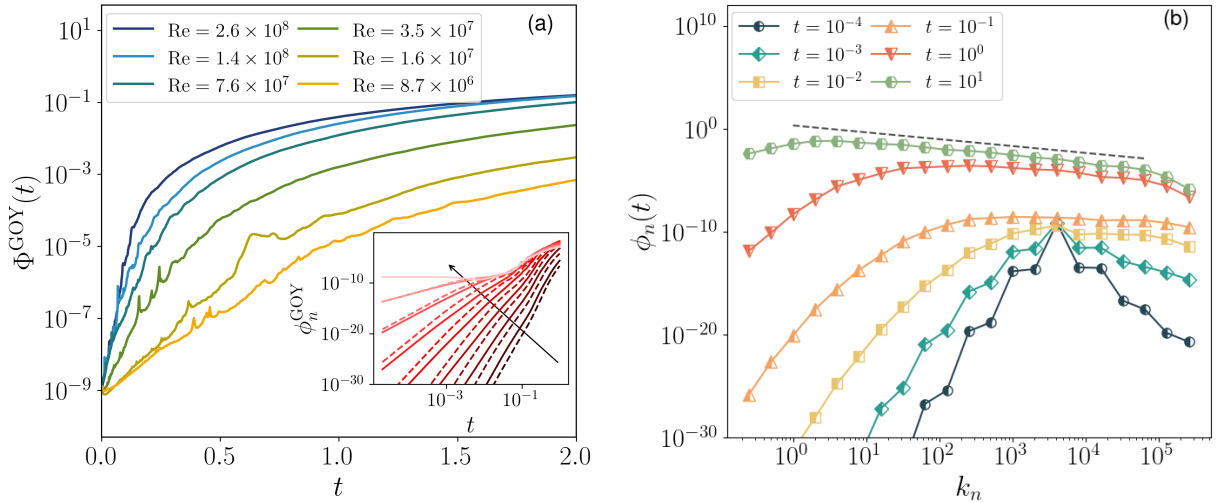


Figure 2: (a) In the left panel, semilog plot of the shell model decorrelator Φ^{GOY} vs t for different Re showing an exponential growth is illustrated. (Inset) Loglog plot of the shell-specific decorrelator ϕ_n for different shells (the arrow indicates the direction $n = 1$ to $n = 16$ shells) at *very* early times shows a power-law growth. (b) In the right panel, the evolution of ϕ_n vs k_n is shown for the GOY shell model, on a loglog scale at different times. Initially the decorrelator spreads from the perturbation shell to the other shell. In long time limit it shows an asymptotic scaling corresponding to $k_n^{-2/3}$ indicated by the dashed line.

of $n_p = 16$ ensures that the tiny perturbation (with $\epsilon_0 = 10^{-3}$) are always added at wavenumbers just before the onset of the dissipation scales for all the Reynolds number being considered. (These results are robust to the specific choice of n_p or variations in ϵ_0 .)

I now use the nearly-identical fields u_n^A and u_n^B as initial conditions and evolve the GOY shell model independently for systems A and B and hence construct the shell model decorrelator $\phi_n(t) \equiv \langle |u_n^A - u_n^B|^2 / 2 \rangle$. The $\langle \cdot \rangle$ denote an ensemble averaging over independent realisations of the initial field u_n^A . While it is straightforward to write the evolution equation for ϕ_n (discussed later), we will find this is neither revealing nor analytically tractable. Instead, it is useful to work with the numerically constructed ϕ_n and assess its behaviour from our simulations.

Given the local nature of interactions in the shell model[23], it is reasonable to expect that decorrelators such as ϕ_n may well have a self-similar short time growth followed by an eventual exponential rise with a saturation as $t \rightarrow \infty$. In the inset of Fig. 2 a loglog plot of the temporal evolution of $\phi_n(\sim t^{g_n})$ is shown for a small initial time window $0 \leq t \lesssim 0.5$ with a clear power-law range for $n < n_p$. This power-law eventually ($t \gtrsim 0.1$) gives way to the exponential-growth phase of decorrelators. The Lyapunov exponent characterising such shell-by-shell decorrelators ϕ_n are of course n -dependent. In the left panel of Fig 3, I have shown the dependence of the power of exponents g_n and the shell-wise Lyapunov exponent λ_n over shell number n .

Hence we find it useful to construct the n -independent Lyapunov exponent λ^{GOY} via the integrated decorrelator $\Phi^{\text{GOY}} = \sum_1^N \phi_n$. In the main panel of Fig. 2 we show a semilog plot of the temporal evolution of Φ^{GOY} for different values of Re showing, yet again, a clear exponential growth, with a shell

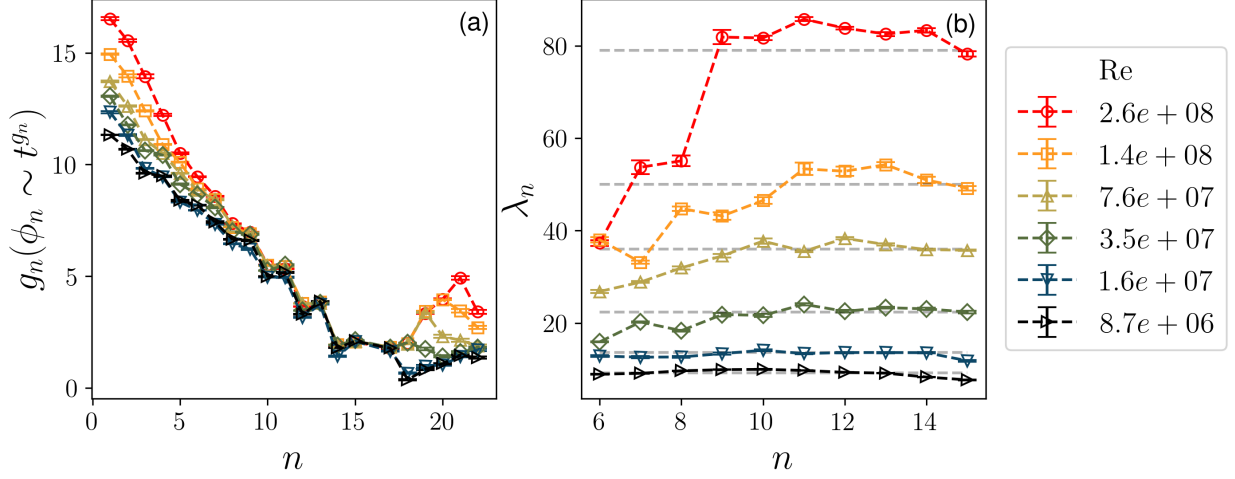


Figure 3: (a) The dependence of the power-law exponents g_n and the shell-wise Lyapunov exponents λ_n over shell number n . Near the perturbation shell number the power-law growths are almost identical for different Re . (b) In the right plot, the dashed line indicates the overall Lyapunov exponents estimated from the summed-decorrelator. Far away from the perturbation shell number, the exponential growth is valid over a very small time window and it is hard to obtain a trustable exponent.

model Lyapunov exponent λ^{GOY} , followed by a saturation to values roughly corresponding to $2E_0$, with E_0 the statistically steady state energy in our shell model.

III.2 Numerical schemes for decorrelator in shell models

The Gledzer-Ohkitani-Yamada or the GOY shell model for three dimensional turbulence

$$\begin{aligned} \frac{d}{dt}u_n &= \iota k_n [u_{n+2}u_{n+1} - \frac{1}{4}u_{n+1}u_{n-1} - \frac{1}{8}u_{n-1}u_{n-2}]^* \\ &- \nu k_n^2 u_n + f_n \end{aligned} \quad (9)$$

is one of several cascade models which mimic Navier-Stokes turbulence and are designed to achieve very high Reynolds number. In such models, the velocities u_n (with boundary conditions $u_{-1} = u_0 = u_{N+1} = u_{N+2} = 0$) associated with a scalar, exponentially growing, wave-number $k_n = k_0 2^n$, are complex, dynamical

variables. The shell numbers range from $1 \leq n \leq N$; in our simulations we choose $N = 22$ and the coefficient of viscosity varies as $2 \times 10^{-7} \leq \nu \leq 64 \times 10^{-7}$ yielding Reynolds numbers in the range $8.7 \times 10^6 \leq \text{Re} \leq 2.6 \times 10^8$; the shell model, Reynolds number is simply defined as $\text{Re} = |u_{\text{rms}}|/k_0\nu$. The equation is integrated numerically by using a fourth-order Runge-Kutta scheme with a time step $\delta t = 2 \times 10^{-5}$. The constant amplitude forcing $f = 0.1$, applied on the $n = 2$ shell and drives the system to a statistically steady state.

We first obtain a non-equilibrium statistically steady state with an energy spectrum $E(k_n) \equiv |u_n|^2/k_n \sim k_n^{-5/3}$ (as shown in the last section) over an inertial range of shells whose extent is determined by Re . We use this steady state velocity field to define the initial field for system A — $u_{n,0}^A$ and construct the initial condition for system B via $u_{n,0}^B = (1 + \epsilon_0)\delta_{n,n_p}u_{n,0}^A$. In all our simulations we have used $\epsilon_0 = 0.001$ and $n_p = 16$. Numerically it has been seen that the shell wise decorrelators first show a power-law rise later followed by the exponential-growth regime. The initial power law largely depends on the distance from the perturbed shell. Therefore, we add the perturbation in a fixed shell $n_p = 16$ for all values of Re .

III.3 Analytical expression

It is possible to construct an evolution equation for the decorrelator in our shell model but, as we shall see, its structure is not transparent and it is intuitively difficult to understand.

By subtracting the GOY model equations for systems A and B we obtain

the evolution of the shell-by-shell velocity difference $\delta u_n = u_n^B - u_n^A$:

$$\begin{aligned}
\frac{d}{dt}\delta u_n &= \iota k_n [(\delta u_{n+2} u_{n+1}^A + u_{n+2}^A \delta u_{n+1} + \delta u_{n+2} \delta u_{n+1}) \\
&\quad - \frac{\epsilon}{\lambda} (\delta u_{n+1} u_{n-1}^A + u_{n+1}^A \delta u_{n-1} + \delta u_{n+1} \delta u_{n-1}) \\
&\quad + \frac{(\epsilon - 1)}{\lambda^2} (\delta u_{n-2} u_{n-1}^A + \delta u_{n-1} u_{n-2}^A + \delta u_{n-2} \delta u_{n-1})]^* \\
&\quad - \nu k_n^2 \delta u_n.
\end{aligned} \tag{10}$$

From this by suitably multiplying with the complex conjugate field δu^* , the decorrelator obeys

$$\begin{aligned}
\frac{d}{dt}\phi_n &= \frac{\iota k_n}{2} [(\delta u_{n+2} u_{n+1}^A \delta u_n + u_{n+2}^A \delta u_{n+1} \delta u_n + \delta u_{n+2} \delta u_{n+1} \delta u_n) \\
&\quad - \frac{\epsilon}{\lambda} (\delta u_{n+1} \delta u_n u_{n-1}^A + u_{n+1}^A \delta u_{n-1} \delta u_n + \delta u_{n+1} \delta u_n \delta u_{n-1}) \\
&\quad + \frac{(\epsilon - 1)}{\lambda^2} (\delta u_{n-2} u_{n-1}^A \delta u_n + u_{n-2}^A \delta u_{n-1} \delta u_n + \delta u_{n-2} \delta u_{n-1} \delta u_n)]^* \\
&\quad + c.c. - 2\nu k_n^2 \phi_n.
\end{aligned} \tag{11}$$

A summation over all shells leads to simplifications in the evolution of the GOY model decorrelator Φ^{GOY}

$$\begin{aligned}
\frac{d}{dt}\Phi^{\text{GOY}} &= \frac{\iota}{2} \sum k_n [\epsilon \delta u_n u_{n+1}^A \delta u_{n+2} \\
&\quad + (1 - \epsilon) \delta u_n \delta u_{n+1} u_{n+2}^A - u_n^A \delta u_{n+1} \delta u_{n+2}]^* \\
&\quad + c.c. - 2\nu \sum_n k_n^2 \phi_n
\end{aligned} \tag{12}$$

Further progress is only possible through a numerical solution of the decorrelator equation as I have already mentioned earlier.

IV Many-body chaos in the Navier-Stokes equation

We first take the three-dimensional, Navier-Stokes equation for an incompressible, unit-density fluid, driven by an external force to a non-equilibrium statistically steady state (NESS). The force $\mathbf{f}$ varies to ensure a constant energy injection [31] $\mathcal{I} = \langle \mathbf{f} \cdot \mathbf{u} \rangle \equiv \overline{\langle \epsilon \rangle}$, where the mean energy dissipation rate $\overline{\langle \epsilon \rangle}$ is the time $\overline{\cdots}$ and space $\langle \cdots \rangle$ average of the spatio-temporally fluctuating dissipation $\epsilon(\mathbf{x}, t)$ characteristic of such non-equilibrium steady states of fully developed turbulence with a statistically steady velocity field $\mathbf{u}_{\text{steady}}$. This follows from the familiar, instantaneous (spatially-averaged) kinetic energy E balance: $\frac{dE}{dt} = -\langle \epsilon \rangle + \mathcal{I}$ with a mean kinetic energy $E_0 = \overline{E}$.

We now construct two, *nearly* identical, velocity fields $\mathbf{u}_0^A \equiv \mathbf{u}_{\text{steady}}$ and $\mathbf{u}_0^B = \mathbf{u}_0^A + \delta\mathbf{u}_0$, where the form of $\delta\mathbf{u}_0$ ensures that system B remains incompressible. We choose the amplitude of perturbation ϵ_0 sufficiently small and localise it in space at $\mathbf{x}_0$; the precise choice of $\mathbf{x}_0$ is irrelevant and can be chosen to be the center of our domain.

We choose these fields — $\mathbf{u}_0^A$ and $\mathbf{u}_0^B$ — as initial conditions at $t = 0$ for the Navier-Stokes equation, evolve them independently in time, and calculate the velocity difference $\delta\mathbf{u}(\mathbf{x}, t) \equiv \mathbf{u}^B(\mathbf{x}, t) - \mathbf{u}^A(\mathbf{x}, t)$ for $t > 0$. From such velocity differences, we are able to construct the decorrelator $\phi(\mathbf{x}, t) \equiv \frac{1}{2}|\delta\mathbf{u}(\mathbf{x}, t)|^2$ [21]. It is easy to show, starting from the Navier-Stokes equation written separately for systems A and B, that the decorrelator follows an evolution (written in component form and sum over repeated indices assumed)

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = \partial_i W_i - \delta u_i S_{ij} \delta u_j + \nu \delta u_i \nabla^2 \delta u_i \quad (13)$$

with the strain-rate tensor $S_{ij} = \frac{1}{2}(\partial_i u_j^A + \partial_j u_i^A)$,

$W_i = u_i^B \phi + \delta u_i \partial_{jk} \int d^3 \mathbf{x}' G(\mathbf{x}, \mathbf{x}') [u_j^A \delta u_k + u_k^A \delta u_j + \delta u_j \delta u_k]'$, where $G(\mathbf{x}, \mathbf{x}')$ is the Green's function obtained from the Poisson equation for pressure, and ν the coefficient of kinematic viscosity.

A further simplification exploits the statistical isotropy and homogeneity of fully developed turbulence to construct, over the volume $\mathcal{V}$, the spatially integrated decorrelator $\Phi(t) \equiv \frac{1}{\mathcal{V}} \int d\mathbf{x} \phi(\mathbf{x}, t) = \langle \phi \rangle$. This leads to a simple cancellation of the divergence term in Eq. (13) and a resulting evolution equation for the integrated decorrelator:

$$\dot{\Phi} = \frac{d\Phi}{dt} = \beta_S + \beta_\eta, \quad (14)$$

where $\beta_S = -\langle \delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u} \rangle$ and $\beta_\eta = \nu \langle \delta \mathbf{u} \cdot \nabla^2 \delta \mathbf{u} \rangle$ as the contributions from the strain and dissipative terms, respectively (The derivation of equation (13) is explicitly explained later). There is one additional remark to make. In the derivations of Eqs. (13)-(14) we have neglected the forcing terms on systems A and B. This is perfectly valid at short times but in the long time $t \rightarrow \infty$ limit it plays a critical role as we shall see below.

At short times we conjecture an exponential growth of the decorrelator: $\dot{\Phi}/\Phi \equiv \lambda$. We confirm this numerically by integrating Eq. (14) with an initial perturbation field $\delta \mathbf{u}$ and measurements of the strain-rate tensor $\mathbf{S}$ drawn from direct numerical simulations (DNSs) of the Navier-Stokes equation (Appendix A)

In the left panel of Fig. 4 we show a representative plot of $\dot{\Phi}/\Phi$ vs time for $\text{Re} = 178.5$ displaying a clear plateau λ demarcated by a pair of vertical dashed lines. While this is not very surprising, given that turbulence is chaotic, it is far from obvious how to trace its origins in the structure of Eq. (14) and in particular how the competing effects of the strain and viscous terms

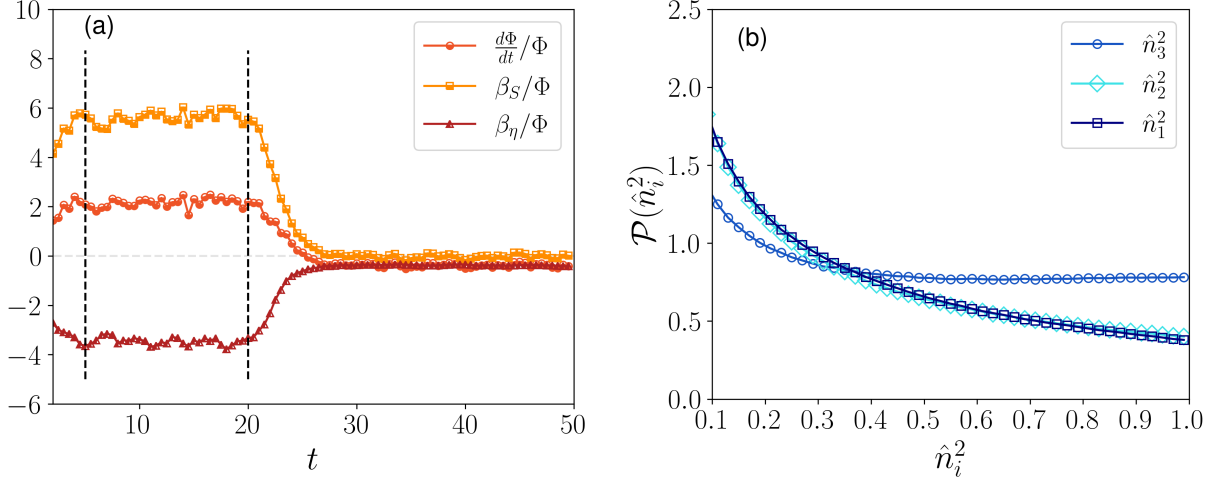


Figure 4: (a) Representative plots of the normalised spatially averaged decorrelator $\dot{\Phi}/\Phi$, β_S/Φ and β_η/Φ vs time for $Re = 178.5$. The pair of vertical dashed lines indicate the short-time exponential-growth phase. (b) In the right panel, the probability density function of the $\hat{n}_i^2$ direction cosines of $\mathbf{u}$ (at short times) is shown. It shows a preferential growth of the difference field along the compressional $i = 3$ direction.

conspire to give such exponential-growth phases. In Fig. 4 we plot β_S and β_η (compensated by Φ) and find strong evidence of their short-time exponential growth — the plateau between the vertical dashed lines — corresponding to Lyapunov exponents $\tilde{\lambda}_S (= \beta_S/\Phi)$ and $\tilde{\lambda}_\eta (= \beta_\eta/\Phi)$, respectively. Furthermore, the dissipative effect of the viscous term $\tilde{\lambda}_\eta \sim \mathcal{O}(1/\tau_\eta) < 0$ is compensated by the effect of the strain term $\tilde{\lambda}_S > |\lambda_\eta|$, leading to an overall positive Lyapunov exponent $\lambda = \tilde{\lambda}_S + \tilde{\lambda}_\eta$.

How do we understand the origins of $\tilde{\lambda}_S$ which leads to a compensation of the negative $\tilde{\lambda}_\eta$ and hence a positive λ ? In particular, how does the strain field S ensure the exponential amplification of $\delta\mathbf{u}$ at short times and thence $\Phi \sim e^{\lambda t}$? The answer lies in decomposing the strain term in the eigenbasis of the strain tensor $\langle \delta\mathbf{u} \cdot \mathbf{S} \cdot \delta\mathbf{u} \rangle = \sum_{i=1}^3 \langle \hat{n}_i^2 \gamma_i |\delta\mathbf{u}|^2 \rangle$, with $\hat{n}_i$ direction cosines of $\mathbf{u}$ (along the eigendirections) and γ_i the eigenvalues. Clearly the dominant contribution in this expansion *must* come from the compressional eigendirection $\langle \delta\mathbf{u} \cdot \mathbf{S} \cdot \delta\mathbf{u} \rangle \approx \langle \hat{n}_3^2 \gamma_3 |\delta\mathbf{u}|^2 \rangle$ and hence $\tilde{\lambda}_S \sim -\gamma_3 \sim \mathcal{O}(1/\tau_\eta) > 0$. We confirm

this preferential stretching along the compressional direction by calculating the probability distribution functions of the three eigendirections $\hat{n}_i$. The relative distributions for the extensional ($i = 1$; $\gamma_1 > 0$), intermediate ($i = 2$; $\gamma_2 \approx 0$) and compressional directions ($i = 3$; $\gamma_3 < 0$), seen in the right panel of Fig. 4, underlines this bias.

In the long time limit, $\beta_S \rightarrow 0$, all eigenvalues are sampled equally Fig 6 leading to $\langle \delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u} \rangle \propto \sum_{i=1}^3 \gamma_i = 0$ because of incompressibility. We see from Fig. 4 that this is indeed the case as the compensated β_S term rapidly falls to 0 for $t \gtrsim 25$.

Does the viscous term β_η also go to zero similarly at late times leading to $\dot{\Phi} = 0$? At such late times it is easy to show that (Derived later) $\beta_\eta \rightarrow -2\overline{\langle \epsilon \rangle}$. Indeed, in Fig. 4 we do see a clear saturation of the normalised viscous term to a value consistent with $-2\overline{\langle \epsilon \rangle} \approx 1.6$ (corresponding to the simulations from which the rate of strain matrix is drawn).

A trivial consequence of this is that within the framework of Eq. (14) $\dot{\Phi} = -2\overline{\langle \epsilon \rangle}$ (clearly seen in Fig. 4) and suggesting that the decorrelator never saturates. However this is a contradiction: By definition, the decorrelator is defined as the spatial average of the velocity difference of systems A and B, and hence at long times $\Phi \rightarrow 2E_0$ since the cross-correlator $\langle \mathbf{u}_A \cdot \mathbf{u}_B \rangle$ vanishes as the fields decorrelate as $t \rightarrow \infty$.

This contradiction is resolved by recalling that the derivation of Eq. (14) neglects the effective energy injection. While at short times, this is zero, at long times this contribution is no longer irrelevant. In fact it can be shown (derived later) that $\langle \delta \mathbf{f} \cdot \delta \mathbf{u} \rangle = 2\overline{\langle \epsilon \rangle}$ as $t \rightarrow \infty$ and thus compensates the viscous contribution $-2\overline{\langle \epsilon \rangle}$ leading to $\dot{\Phi} = 0$.

How consistent are these ideas in actual measurements of the decorrelator

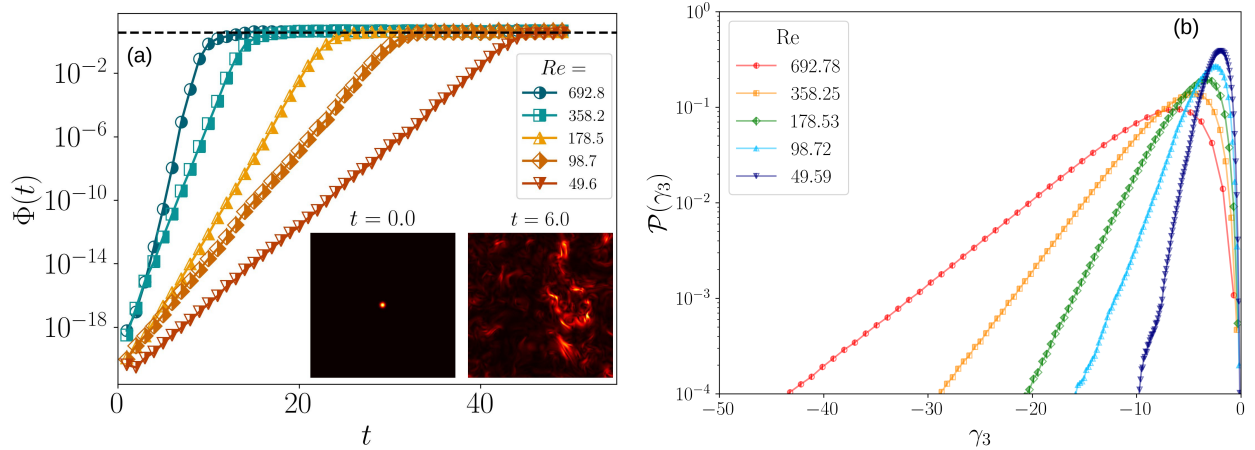


Figure 5: (a) Semilog plots of Φ vs t from DNSs for different Re showing an exponential regime followed by saturation at a value $2E_0$ (dashed horizontal line). The insets show a 2D slice ($Z = \pi$) at (left) early and (right) late times of the square of the difference field $|\delta \mathbf{u}|^2$ for $Re = 178.5$. In the right (b) we have the probability distribution of γ_3 which shows both peak and the spread of the distribution scales with Re

in DNSs of the Navier-Stokes equation? We check for this by solving the incompressible Navier-Stokes equation on a 2π triply-periodic domain by using a pseudospectral method with a large-scale constant energy $\mathcal{I} = \overline{\langle \epsilon \rangle}$ [31] injection scheme to drive the flow and maintain it in non equilibrium statistically steady state with a mean energy E_0 ; see Appendix A for a detailed summary of the numerical scheme and the parameters used. By using the same protocol described earlier, we calculate the decorrelator Φ for several different values of the Reynolds number Re . In Fig. 5 we show representative snapshots of the two-dimensional slice (over the XY plane; $Z = \pi$) at early (left inset) and late (right inset) times, corresponding to exponential-growth and decorrelated phases, respectively.

These difference fields allow us to construct the spatially averaged decorrelator Φ which we plot, on a semilog scale, in the main panel of Fig. 5 as a function of time. We see a convincing exponential regime from which we can extract the Lyapunov exponent λ^{DNS} , followed by a saturation at a value approximately equal to $2E_0$, indicated by the dashed horizontal line, as our theory

suggests. (We have confirmed from our data that the onset of the saturation of the decorrelators (see Figs. 4 and 5) is of the order of the inverse of the Lyapunov exponent.)

It is important to observe that even at very short times, before the exponential growth phase, our results from the Navier-Stokes equation summarised in Fig. 5 do not suggest an initial power-law growth which has been seen in many local models of chaotic systems[23, 28]. Could this — as well as the somewhat self-similar evolution of the difference field (see Fig. 5, insets) — be a consequence of the essential non-local nature of the equation?

V Decorrelator from the Navier-Stokes equation and its limits

In this section, we provide the derivation of the equation of motion of the decorrelator for the Navier-Stokes equation.

We begin by recalling that the pressure term in the Navier-Stokes equation can be rewritten as a Poisson equation

$$\nabla^2 P = -\partial_j u_i \partial_i u_j \quad (15)$$

by exploiting the incompressibility constraint. This Poisson equation underlines the non-local nature of turbulence. It is useful, in what follows, to have a Green's function representation —

$$P(\mathbf{x}) = - \int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_j u_i \partial_i u_j]' \quad (16)$$

where the $'$ denotes quantities are at $\mathbf{x}'$.

We use this representation of the pressure field to construct the evolution equation of the velocity perturbation $\delta \mathbf{u} = \mathbf{u}^B - \mathbf{u}^A$

$$\begin{aligned} \frac{\partial \delta u_i}{\partial t} &= -\delta u_j \partial_j u_i^A - u_j^B \partial_j \delta u_i \\ &\quad - \partial_i \left(\int d\mathbf{x}' G(\mathbf{x}, \mathbf{x}') [\partial_{jk}^2 (u_j^A \delta u_k + \delta u_j u_k^A + \delta u_j \delta u_k)]' \right) \\ &\quad + \nu \nabla^2 \delta u_i + \delta u_i \delta f_i \end{aligned} \quad (17)$$

by subtracting the Navier-Stokes equation written for systems A and B from each other and keeping in mind that $\delta \mathbf{u}$ is solenoidal. A further dot product

between Eq 17 and $\delta \mathbf{u}$ leads to

$$\frac{\partial \phi}{\partial t} = -\delta u_i S_{ij} \delta u_j - \partial_i W_i + \nu \delta u_i \nabla^2 \delta u_i + \delta u_i \delta f_i \quad (18)$$

where $S_{ij} = \frac{1}{2}(\partial_j u_i^A + \partial_i u_j^A)$ is the strain-rate tensor and

$$W_i = u_i^B \phi + \delta u_i \int d\mathbf{x} G(\mathbf{x}, \mathbf{x}') [\partial_{jk}^2 (u_j^A \delta u_k + \delta u_j u_k^A + \delta u_j \delta u_k)]'$$

A spatial integration with $\Phi = \int d\mathbf{x} \phi$ now leads to the cancellation of the divergence term and hence

$$\frac{d\Phi}{dt} = -\langle \delta u_i S_{ij} \delta u_j \rangle + \nu \langle \delta u_i \nabla^2 \delta u_i \rangle + \langle \delta u_i \delta f_i \rangle \equiv \beta_S + \beta_\eta + \langle \delta u_i \delta f_i \rangle \quad (19)$$

the evolution equation for our spatially integrated decorrelator as discussed in the last chapter.

At early times, before systems A and B decorrelate completely, the forcing terms are more or less identical for both systems and hence in this exponential growth phase $\langle \delta u_i \delta f_i \rangle = 0$. At long times, beyond this exponential-growth phase when the systems have started to completely decorrelate this is not longer valid. Thus as $t \rightarrow \infty$, we estimate $\langle \delta u_i \delta f_i \rangle = \langle u_i^B f_i^B \rangle + \langle u_i^A f_i^A \rangle - \langle (u_i^B f_i^A + u_i^A f_i^B) \rangle \rightarrow 2\overline{\langle \epsilon \rangle}$. As we shall see below, this exactly compensates the contribution of the viscous term β_η at long times.

The viscous term can be expanded as

$$\begin{aligned} \beta_\eta &= \nu \langle \delta u_i \nabla^2 \delta u_i \rangle = -\nu \langle \partial_j \delta u_i \partial_j \delta u_i \rangle \\ &= -\nu \langle (\partial_i u_j^B \partial_i u_j^B + \partial_i u_j^A \partial_i u_j^A - 2\partial_i u_j^B \partial_i u_j^A) \rangle \\ &= -4\nu \text{Tr}(S^2) + 2\nu \langle \partial_i u_j^A \partial_i u_j^B \rangle = 2\langle \epsilon \rangle + 2\nu \langle \partial_i u_j^A \partial_i u_j^B \rangle. \end{aligned} \quad (20)$$

We find at short times, in the exponential-growth phase this terms scales as $\beta_\eta = \mathcal{O}(1/\tau_\eta)$, where $\tau_\eta = \sqrt{\frac{\nu}{\langle \epsilon \rangle}}$ corresponds to the smallest Kolmogorov time

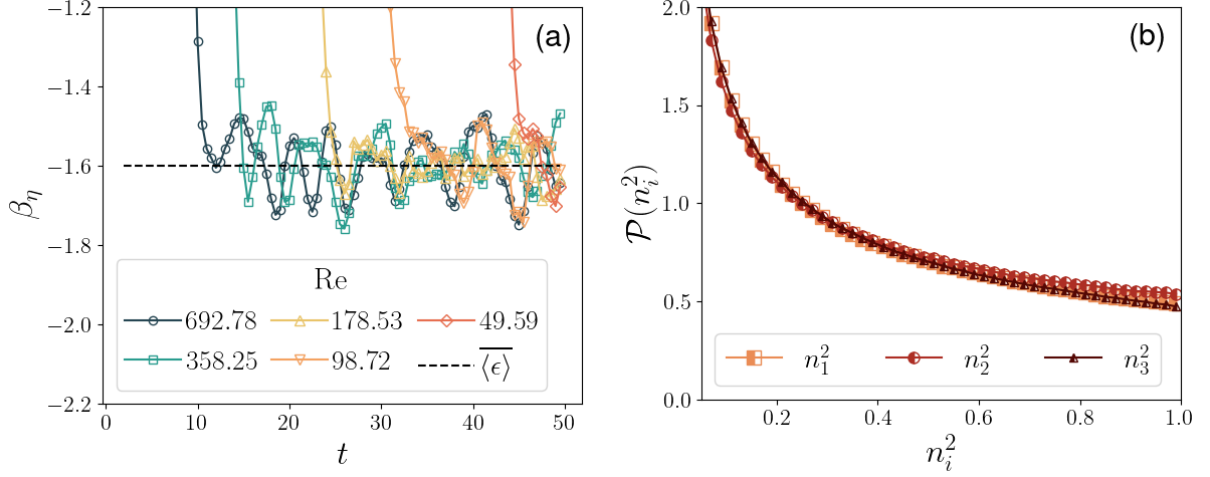


Figure 6: (a) Plots of β_η , at late times, for different Reynolds numbers, showing a saturation to $-2\langle\epsilon\rangle$ (black dotted, horizontal line). (b) A plot of the probability distribution of components of $\delta\mathbf{u}$ (for $\text{Re} = 358.2$), for the Navier-Stokes equation, along the three eigen vectors of the strain rate tensor at late times once the decorrelator saturates.

scales. However, at long times $t \rightarrow \infty$, systems A and B decorrelate leading to $\langle\partial_i u_j^A \partial_i u_j^B\rangle \rightarrow 0$ and hence $\beta_\eta \sim -2\langle\epsilon\rangle$ (see Fig. 6(a) as well as the main text). This is easily understandable because on average dissipation is balanced by the energy injection derived above.

We now turn to the strain term β_S . At short times, as we have seen in the main text, this has a contribution which compensates, with the perturbation field aligning along the compressional eigenvector, the negative contribution of β_η leading to an overall exponential growth of the decorrelator in the exponential-growth phase. However, this preferential alignment disappears, as $t \rightarrow \infty$, since the perturbed velocity field samples all the eigenvalues with equal probability and hence, because of the incompressibility condition, the strain rate $\beta_S \rightarrow 0$ as discussed in the main text. In Fig. 6(b) we show how, as $t \rightarrow \infty$ and systems A and B decorrelate, the equal sampling of the 3 eigendirections of the strain-rate tensor in contrast to its preferential alignment at times corresponding to the exponential-growth regime. Typically the decorrelator saturates over a time-scale $\sim \mathcal{O}(\lambda^{-1})$.

VI Conclusions

We can now summarise all of these ideas in answering the question of how consistent are the various (largest) Lyapunov exponent — λ , λ^{DNS} , and λ^{GOY} — with each other given the different ways and indeed the different models from which they are derived? In Fig. 7 we show a composite plot of λ , λ^{DNS} and λ^{GOY} as well as $|\gamma_3|$, versus the Reynolds number Re . For measurements from the DNSs and the GOY shell model simulations, we estimate the Lyapunov exponents from a linear fit in the exponential-growth phase of the decorrelators (as shown in Figs. 5 and 2); the errors on such fits yield the errorbars on λ^{DNS} and λ^{GOY} . We find our errorbars smaller than the symbols sizes in Fig. 7. The errorbars for the exponent γ_3 are extracted likewise and are of the same order as those on λ^{DNS} and λ^{GOY} . The exponent λ is obtained differently. We use plots of $\dot{\Phi}/\Phi$, such as the one shown in Fig. 4, to extract λ and its errorbar as

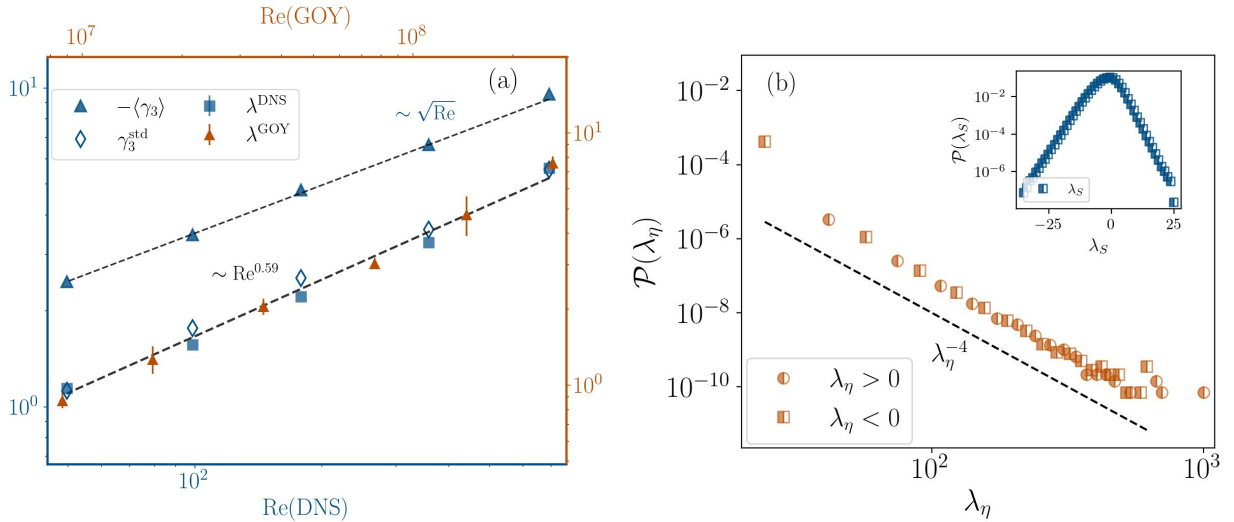


Figure 7: (a) A loglog plot of λ , λ^{DNS} , γ_3^{std} and $-\langle\gamma_3\rangle$, obtained from various DNSs versus the Reynolds number Re (DNS) (bottom X axis), as well as the Lyapunov exponent obtain from the shell model λ^{GOY} (rescaled by a constant factor to overlap with the DNS data since the Reynolds number ranges are different) with the corresponding shell model Reynolds numbers $\text{Re}(\text{GOY})$ (top X axis). (b) A loglog plot of the probability density function $\mathcal{P}$ for λ_η showing a power-law $|\lambda_\eta|^{-4}$ tail. (Inset) the exponential tail of the probability distribution $\mathcal{P}$ of λ_S is plotted.

the mean and standard deviation of the plateau, respectively.

We find remarkable consistency (within error bars) between the different measurements of the Lyapunov exponents and indeed across the different models, as well as the range of Reynolds number between the DNSs (bottom axis) and shell models (top axis), that we use. All the Lyapunov exponents seem to follow a universal scaling: $\lambda^{\text{DNS}} = \lambda^{\text{GOY}} = \lambda \sim \text{Re}^\alpha$, with $\alpha = 0.59 \pm 0.04$.

The theory developed earlier suggests a connection between the Lyapunov exponent and the compressional eigenvalue γ_3 . Given the departure from of the measured Lyapunov exponents (Fig. 7) from a Kolmogorov-like mean field exponent of 0.5, it is reasonable to conjecture that this deviation stems from the intermittent fluctuations which characterise fully developed turbulence. The corollary to this is that the standard deviation γ_3^{std} of γ_3 , reflecting such fluctuations, out to scale with the Reynolds number with the same exponent α as do the Lyapunov exponents. In Fig. 7 we show a plot of γ_3^{std} from our DNSs and find a near perfect agreement with the Lyapunov exponent. Furthermore, we confirm that the mean eigenvalue $-\langle\gamma_3\rangle \sim \sqrt{Re}$ (Fig. 7) consistent with the simpler scaling analysis not accounting for fluctuations.

In this work we have shown how recent ideas of decorrelators [19, 20, 18, 32, 21] provide a microscopic way to understand the origins of chaos in fully developed turbulence. Remarkably, these ideas are just as robust for simpler, cascade models of turbulence where locality [23] ensures an initial scale-invariant growth of the decorrelator. Remarkably, are able to factor in the intermittent fluctuations of the velocity-gradient tensor to show how the standard deviation of γ_3 leads to $\alpha = 0.59 \pm 0.04$ 0.5. We find signatures of these fluctuations in one further aspect of the strain and viscous terms in Eq. (14) which connects with the idea of intermittency. The theory developed by us involve the *mean*

exponents: λ_S and λ_η (and hence λ) are obtained from the spatial integrals over the corresponding terms. However, it is possible to measure the *local* exponents $\lambda_S(\mathbf{x}) \equiv \frac{\delta \mathbf{u} \cdot \mathbf{S} \cdot \delta \mathbf{u}}{|\delta \mathbf{u}|^2/2}$ and $\lambda_\eta(\mathbf{x}) \equiv \frac{\nu \delta \mathbf{u} \cdot \nabla^2 \delta \mathbf{u}}{|\delta \mathbf{u}|^2/2}$ (whose mean values are indeed λ_S and λ_η) in the exponential-growth phase and thence their probability distribution functions $\mathcal{P}$. These distributions are found to have exponential (for $\lambda_S(\mathbf{x})$) or power-law (for $\lambda_\eta(\mathbf{x})$; see inset of Fig. 7) tails strongly suggestive of intermittency and its connections with the question of chaos in turbulence. We do not explore these ideas, and in particular the origins of the power-law tails, any further in this work as well as its implications for Lagrangian chaos [13, 33], but leave it for a more detailed study of the statistics of β_S and β_η in the future.

VII Appendix

VII.1 Direct Numerical Simulation of the N.S. equation

We perform direct numerical simulations (DNSs), by using a pseudospectral method, of the three-dimensional (3D), incompressible, unit density Navier Stokes equation, with the pressure field P

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u} - \nabla P + \mathbf{f}. \quad (21)$$

on a periodic box of size $L = 2\pi$ with N^3 collocation points. We choose $N = 256$ and $N = 512$ (to check for numerical convergence) and vary the coefficient viscosity $2 \times 10^{-3} \leq \nu \leq 32 \times 10^{-3}$ to obtain Reynolds numbers $99 \leq \text{Re} \leq 693$. We use a second-order Adams-Bashforth for time-marching with a time step $\delta t = 5 \times 10^{-4}$ (for $N = 256$) and $\delta t = 4 \times 10^{-4}$ (for $N = 512$).

We initialise the flow with a random initial condition such that the initial energy spectrum $E(k) = A_0 k^2 \exp(-k^2/2k_0^2)$ with $A_0 = k_0 = 1$. The system is driven to a non-equilibrium steady state (NESS) through a large-scale forcing with a constant energy injection at large scales corresponding to wavenumber $1 \leq k_{\text{force}} \leq 2$.

This statistically steady velocity field $\mathbf{u}$ is taken as the initial field for system A: $\mathbf{u}_0^{\text{A}} \equiv \mathbf{u}$ and further define the initial condition for system B via $\mathbf{u}_0^{\text{B}} = \mathbf{u}_0^{\text{A}} + \delta \mathbf{u}_0$. We use a Gaussian perturbation $\delta \mathbf{u}_0(\mathbf{x}) = \epsilon \exp(-(x - x_0)^2/2\sigma_0^2) \hat{\mathbf{x}}$ with $\epsilon = 10^{-5}$, $\sigma_0 = 4dx$ and $x_0 = (\pi, \pi, \pi)$ at the center of the domain. Although initially this field is not divergence free but as we integrate the Navier-Stokes equation using the pseudo-spectral scheme, in time $t = \delta t$, the projection operator projects this as a solenoidal field.

The nearly identical fields $\mathbf{u}_0^A$ and $\mathbf{u}_0^B$ are then used as initial conditions for the Navier-Stokes equation and evolved as before. The decorrelator is calculated from the evolution of such twin simulations as $\phi(\mathbf{x}, t) = |\mathbf{u}^B - \mathbf{u}^A|^2/2$ and thence $\Phi(t) = \int d\mathbf{x} \phi(\mathbf{x}, t)/\mathcal{V}$. The decorrelators are then used to estimate the Lyapunov exponents (obtained by fitting the initial exponential growth phase) and its dependence on the Reynolds number.

VII.2 Numerical Integration Schemes

Next I will briefly describe the numerical schemes that I have used. For the shell model, I have primarily used the fourth order Runge-Kutta method. If we start with a general equation $\dot{y} = f(y, t)$ with an initial condition $y(t = 0) = y_0$, then the time marching is done using the following scheme –

$$\begin{aligned}
k_1 &= \delta t \times f(y_t, t) \\
k_2 &= \delta t \times f\left(y_t + \frac{k_1}{2}, t + \frac{\delta}{2}\right) \\
k_3 &= \delta t \times f\left(y_t + \frac{k_2}{2}, t + \frac{\delta}{2}\right) \\
k_4 &= \delta t \times f(y_t + k_3, t + \delta) \\
y_{t+\delta t} &= y_t + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned} \tag{22}$$

where δt is the discrete time step. The numerical scheme mentioned above is a fourth order method and hence the typical local truncation error is $\mathcal{O}(\delta t^5)$ which essentially allows us a better accuracy.

For the Direct Numerical Simulation of the Navier-Stokes equation I however, used a second order Adam-Bashforth scheme. In the following, I will briefly try to explain the numerical scheme for such a system. Suppose, we

have an differential equation of the form —

$$\frac{dy}{dt} = -\alpha y + f(t)$$

Therefore, we can write —

$$\begin{aligned} \implies \frac{dy}{dt} + \alpha y &= f(t) \\ \implies \frac{dy}{dt} e^{\alpha t} + \alpha y e^{\alpha t} &= f(t) e^{\alpha t} \\ \implies y(t + \delta t) e^{\alpha(t+\delta t)} - y(t) e^{\alpha t} &= \int_t^{t+\delta t} ds e^{\alpha s} f(s) \end{aligned}$$

Similarly, we can obtain (in a backward sense) —

$$y(t - \delta t) e^{\alpha(t-\delta t)} - y(t) e^{\alpha t} = \int_t^{t-\delta t} ds e^{\alpha s} f(s)$$

Therefore, substracting the last two equation from each other we obtain the following relation —

$$y(t + \delta t) = y(t - \delta t) e^{-2\alpha\delta t} + \int_{t-\delta t}^{t+\delta t} ds e^{-\alpha(t+\delta t-s)} f(s) \quad (23)$$

By suitably approximating $f(s)$ as $\frac{3}{2}f(t) - \frac{1}{2}f(t - \delta t)$ (second order) one obtains the Adam-Bashforth numerical integration scheme.

$$y(t + \delta t) = y(t - \delta t) e^{-2\alpha\delta t} + \frac{1 - e^{-2\alpha\delta t}}{\alpha} \left(\frac{3}{2}f(t) - \frac{1}{2}f(t - \delta t) \right) \quad (24)$$

A similar method can also be used for integrating the shell model equation. However, given the numerical simplicity of such a model it is practically possible to easily implement higher order methods (RK4).

References

- [1] Rahul Pandit, Prasad Perlekar, and Samriddhi Sankar Ray. Statistical properties of turbulence: An overview. *Pramana*, 73(1):157–191, 2009. doi: 10.1007/s12043-009-0096-6.
- [2] Uriel Frisch. *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press, 1995.
- [3] James Gleick. *Chaos: Making a New Science*. Viking, New York, 1987. ISBN 978-0-670-81178-6.
- [4] David Ruelle. Microscopic fluctuations and turbulence. *Physics Letters A*, 72(2):81–82, 1979. ISSN 0375-9601. doi: [https://doi.org/10.1016/0375-9601\(79\)90653-4](https://doi.org/10.1016/0375-9601(79)90653-4). URL <https://www.sciencedirect.com/science/article/pii/0375960179906534>.
- [5] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. *Dokl. Akad. Nauk SSSR*, 30:299–303, 1941.
- [6] U. Frisch and G. Parisi. Turbulence and predictability of geophysical fluid dynamics and climate dynamics. *Proceedings of the International School of Physics Enrico Fermi, Course LXXXVIII, Varenna, 1985*, 1985. URL <https://cir.nii.ac.jp/crid/1570291225411264384>.
- [7] A. Crisanti, M. H. Jensen, A. Vulpiani, and G. Paladin. Intermittency and predictability in turbulence. *Phys. Rev. Lett.*, 70:166–169, Jan 1993. doi: 10.1103/PhysRevLett.70.166. URL <https://link.aps.org/doi/10.1103/PhysRevLett.70.166>.
- [8] R Benzi, G Paladin, G Parisi, and A Vulpiani. On the multifractal nature of fully developed turbulence and chaotic systems. *Journal of Physics A: Mathematical and General*, 17(18):3521, dec 1984. doi: 10.1088/0305-4470/17/18/021. URL <https://dx.doi.org/10.1088/0305-4470/17/18/021>.
- [9] Michio Yamada and Koji Ohkitani. Lyapunov spectrum of a chaotic model of three-dimensional turbulence. *Journal of the Physical Society of Japan*, 56(12):4210–4213, 1987.
- [10] Richard D. J. G. Ho, Daniel Clark, and Arjun Berera. Chaotic measures as an alternative to spectral measures for analysing turbulent flow. *Atmosphere*, 15(9), 2024. ISSN 2073-4433. doi: 10.3390/atmos15091053. URL <https://www.mdpi.com/2073-4433/15/9/1053>.

- [11] Itzhak Fouxon, Joshua Feinberg, Petri K  pyl  , and Michael Mond. Reynolds number dependence of lyapunov exponents of turbulence and fluid particles. *Phys. Rev. E*, 103: 033110, Mar 2021. doi: 10.1103/PhysRevE.103.033110. URL <https://link.aps.org/doi/10.1103/PhysRevE.103.033110>.
- [12] Arjun Berera and Richard D. J. G. Ho. Chaotic properties of a turbulent isotropic fluid. *Phys. Rev. Lett.*, 120:024101, Jan 2018. doi: 10.1103/PhysRevLett.120.024101. URL <https://link.aps.org/doi/10.1103/PhysRevLett.120.024101>.
- [13] J  r  mie Bec, Luca Biferale, Guido Boffetta, Massimo Cencini, Stefano Musacchio, and Federico Toschi. Lyapunov exponents of heavy particles in turbulence. *Physics of Fluids*, 18(9):091702, 09 2006. ISSN 1070-6631. doi: 10.1063/1.2349587. URL <https://doi.org/10.1063/1.2349587>.
- [14] Siddhartha Mukherjee, Jer  me Schalkwijk, and Harmen J. J. Jonker. Predictability of dry convective boundary layers: An les study. *Journal of the Atmospheric Sciences*, 73(7): 2715 – 2727, 2016. doi: 10.1175/JAS-D-15-0206.1. URL <https://journals.ametsoc.org/view/journals/atsc/73/7/jas-d-15-0206.1.xml>.
- [15] G. Boffetta and S. Musacchio. Chaos and predictability of homogeneous-isotropic turbulence. *Phys. Rev. Lett.*, 119:054102, Aug 2017. doi: 10.1103/PhysRevLett.119.054102. URL <https://link.aps.org/doi/10.1103/PhysRevLett.119.054102>.
- [16] Prakash Mohan, Nicholas Fitzsimmons, and Robert D. Moser. Scaling of lyapunov exponents in homogeneous isotropic turbulence. *Phys. Rev. Fluids*, 2:114606, Nov 2017. doi: 10.1103/PhysRevFluids.2.114606. URL <https://link.aps.org/doi/10.1103/PhysRevFluids.2.114606>.
- [17] Jin Ge, Joran Rolland, and John Christos Vassilicos. The production of uncertainty in three-dimensional navier–stokes turbulence. *Journal of Fluid Mechanics*, 977:A17, 2023. doi: 10.1017/jfm.2023.967.
- [18] Avijit Das, Saurish Chakrabarty, Abhishek Dhar, Anupam Kundu, David A. Huse, Roderich Moessner, Samriddhi Sankar Ray, and Subhro Bhattacharjee. Light-cone spreading of perturbations and the butterfly effect in a classical spin chain. *Phys. Rev. Lett.*, 121:024101, Jul 2018. doi: 10.1103/PhysRevLett.121.024101. URL <https://link.aps.org/doi/10.1103/PhysRevLett.121.024101>.

- [19] Thomas Bilitewski, Subhro Bhattacharjee, and Roderich Moessner. Temperature dependence of the butterfly effect in a classical many-body system. *Phys. Rev. Lett.*, 121:250602, Dec 2018. doi: 10.1103/PhysRevLett.121.250602. URL <https://link.aps.org/doi/10.1103/PhysRevLett.121.250602>.
- [20] Thomas Bilitewski, Subhro Bhattacharjee, and Roderich Moessner. Classical many-body chaos with and without quasiparticles. *Phys. Rev. B*, 103:174302, May 2021. doi: 10.1103/PhysRevB.103.174302. URL <https://link.aps.org/doi/10.1103/PhysRevB.103.174302>.
- [21] Sugan Durai Murugan, Dheeraj Kumar, Subhro Bhattacharjee, and Samriddhi Sankar Ray. Many-body chaos in thermalized fluids. *Phys. Rev. Lett.*, 127:124501, Sep 2021. doi: 10.1103/PhysRevLett.127.124501. URL <https://link.aps.org/doi/10.1103/PhysRevLett.127.124501>.
- [22] Lewis F Richardson. *Weather prediction by numerical process*. University Press, 1922.
- [23] Luca Biferale. Shell models of energy cascade in turbulence. *Annual Review of Fluid Mechanics*, 35(Volume 35, 2003):441–468, 2003. ISSN 1545-4479. doi: <https://doi.org/10.1146/annurev.fluid.35.101101.161122>. URL <https://www.annualreviews.org/content/journals/10.1146/annurev.fluid.35.101101.161122>.
- [24] Edward N. Lorenz. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, 20(2):130 – 141, 1963. doi: 10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2. URL https://journals.ametsoc.org/view/journals/atsc/20/2/1520-0469_1963_020_0130_dnf_2_0_co_2.xml.
- [25] Eric D. Siggia. Model of intermittency in three-dimensional turbulence. *Phys. Rev. A*, 17:1166–1176, Mar 1978. doi: 10.1103/PhysRevA.17.1166. URL <https://link.aps.org/doi/10.1103/PhysRevA.17.1166>.
- [26] E. B. Gledzer. System of hydrodynamic type admitting two quadratic integrals of motion. *Soviet Physics Doklady*, 18:216, October 1973.
- [27] A. M. Obukhov. On some general characteristics of the equations of the dynamics of the atmosphere. *Izvestiya Akademii Nauk SSSR, Fizika Atmosfery i Okeana*, 7:695–704, 1971. Translated as: *Izvestiya, Atmospheric and Oceanic Physics*.

- [28] Peter D. Ditlevsen. *Turbulence and Shell Models*. Cambridge University Press, Cambridge, UK, oct 2010. ISBN 9780511919251. doi: 10.1017/cbo9780511919251. URL <http://dx.doi.org/10.1017/CB09780511919251>.
- [29] Mehran Kardar. *Statistical Physics of Particles*. Cambridge University Press, 2007.
- [30] Koji Ohkitani and Michio Yamada. Temporal intermittency in the energy cascade process and local lyapunov analysis in fully-developed model turbulence. *Progress of Theoretical Physics*, 81(2):329–341, 02 1989. ISSN 0033-068X. doi: 10.1143/PTP.81.329. URL <https://doi.org/10.1143/PTP.81.329>.
- [31] A. G. Lamorgese, D. A. Caughey, and S. B. Pope. Direct numerical simulation of homogeneous turbulence with hyperviscosity. *Physics of Fluids*, 17(1):015106, 12 2004. ISSN 1070-6631. doi: 10.1063/1.1833415. URL <https://doi.org/10.1063/1.1833415>.
- [32] Stephen H. Shenker and Douglas Stanford. Black holes and the butterfly effect. *Journal of High Energy Physics*, 2014(3), March 2014. ISSN 1029-8479. doi: 10.1007/jhep03(2014)067. URL [http://dx.doi.org/10.1007/JHEP03\(2014\)067](http://dx.doi.org/10.1007/JHEP03(2014)067).
- [33] Samriddhi Sankar Ray. Non-intermittent turbulence: Lagrangian chaos and irreversibility. *Phys. Rev. Fluids*, 3:072601, Jul 2018. URL <https://link.aps.org/doi/10.1103/PhysRevFluids.3.072601>.